Geometric Algorithms

- primitive operations
- convex hull
- closest pair
- voronoi diagram

References:

Algorithms in C (2nd edition), Chapters 24-25

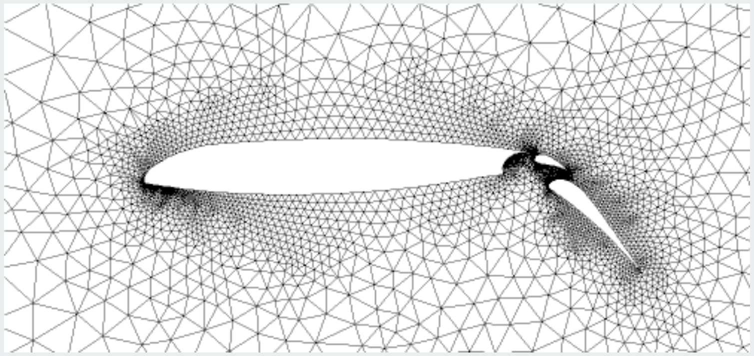
http://www.cs.princeton.edu/introalgsds/71primitives

http://www.cs.princeton.edu/introalgsds/72hull

Geometric Algorithms

Applications.

- Data mining.
- VLSI design.
- Computer vision.
- Mathematical models.
- Astronomical simulation.
- Geographic information systems.
- Computer graphics (movies, games, virtual reality).
- Models of physical world (maps, architecture, medical imaging).



airflow around an aircraft wing

Reference: http://www.ics.uci.edu/~eppstein/geom.html

History.

- Ancient mathematical foundations.
- Most geometric algorithms less than 25 years old.

- primitive operations
- convex hull
- closest pair
- voronoi diagram

Geometric Primitives

Point: two numbers (x, y).

Line: two numbers a and b $[ax + by = 1]$ any line not through origin

Line segment: two points.

Polygon: sequence of points.

Primitive operations.

- Is a point inside a polygon?
- Compare slopes of two lines.
- Distance between two points.
- Do two line segments intersect?
- Given three points p_1, p_2, p_3, is p_1-p_2-p_3 a counterclockwise turn?

Other geometric shapes.

- Triangle, rectangle, circle, sphere, cone, …
- 3D and higher dimensions sometimes more complicated.

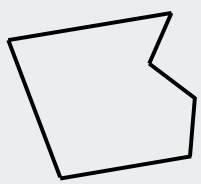
Intuition

Warning: intuition may be misleading.

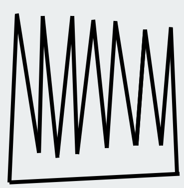
- Humans have spatial intuition in 2D and 3D.
- Computers do not.
- Neither has good intuition in higher dimensions!

Is a given polygon simple?

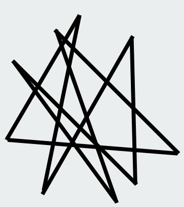
no crossings



1	6	5	8	7	2
7	8	6	4	2	1



1	15	14	13	12	11	10	9	8	7	6	5	4	3	2
1	2	18	4	18	4	19	4	19	4	20	3	20	3	20



1	10	3	7	2	8	8	3	4
6	5	15	1	11	3	14	2	16

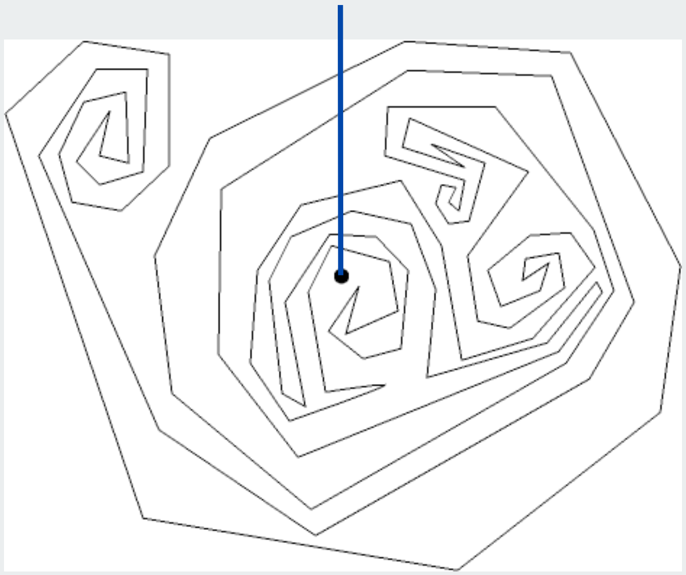
we think of this

algorithm sees this

Polygon Inside, Outside

Jordan curve theorem. [Veblen 1905] Any continuous simple closed curve cuts the plane in exactly two pieces: the inside and the outside.

Is a point inside a simple polygon?



http://www.ics.uci.edu/~eppstein/geom.html

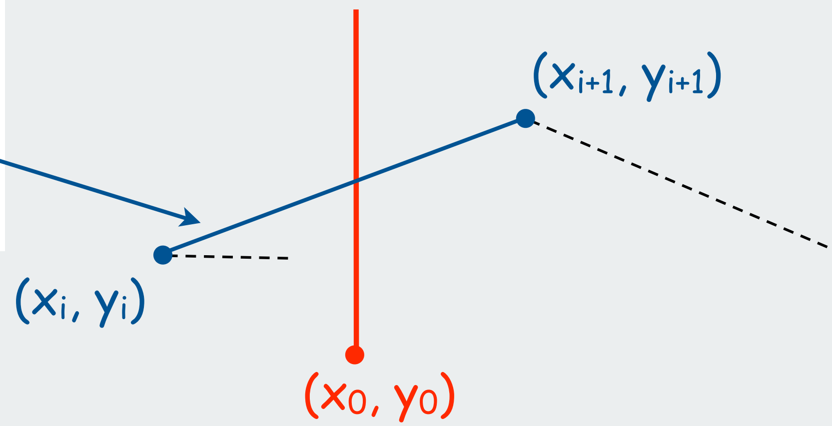
Application. Draw a filled polygon on the screen.

Polygon Inside, Outside: Crossing Number

Does line segment intersect ray?

$$y_0 = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}(x_0 - x_i) + y_i$$

$$x_i \le x_0 \le x_{i+1}$$

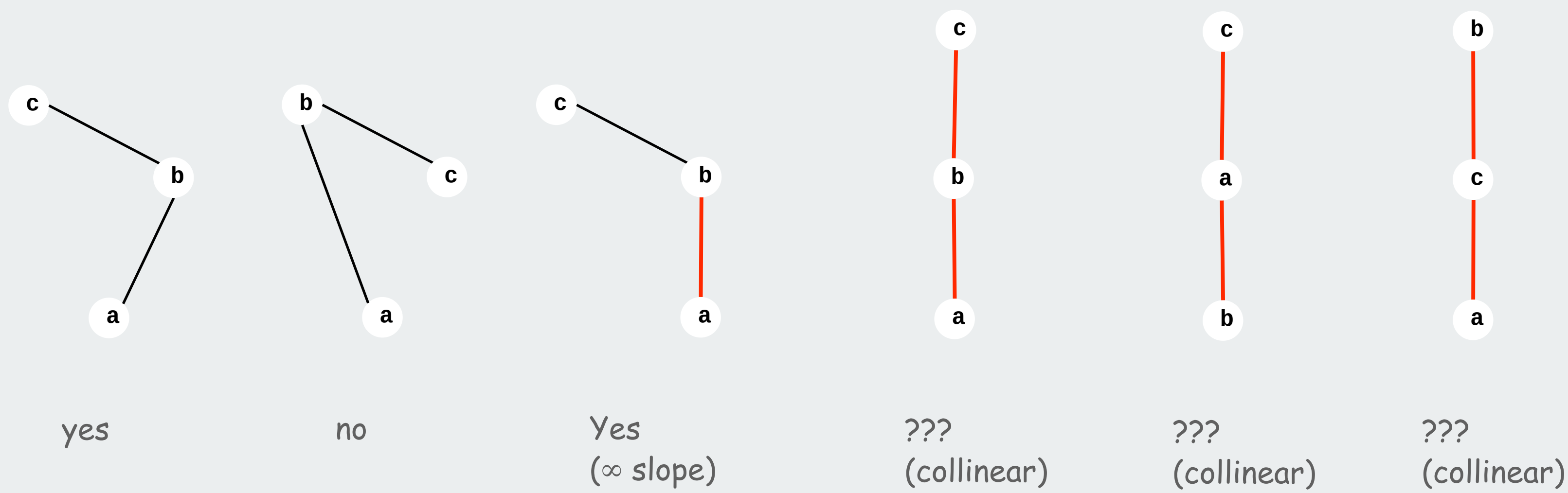



```
public boolean contains(double x0, double y0)
{
    int crossings = 0;
    for (int i = 0; i < N; i++)
    {
       double  slope = (y[i+1] - y[i]) / (x[i+1] - x[i]);
       boolean cond1 = (x[i] <= x0) && (x0 < x[i+1]);
       boolean cond2 = (x[i+1] <= x0) && (x0 < x[i]);
       boolean above = (y0 < slope * (x0 - x[i]) + y[i]);
       if ((cond1 || cond2)  && above ) crossings++;
    }
    return ( crossings % 2 != 0 );
 }
```

Implementing CCW

CCW. Given three point a, b, and c, is a-b-c a counterclockwise turn?

- Analog of comparisons in sorting.
- Idea: compare slopes.



Lesson. Geometric primitives are tricky to implement.

- Dealing with degenerate cases.
- Coping with floating point precision.

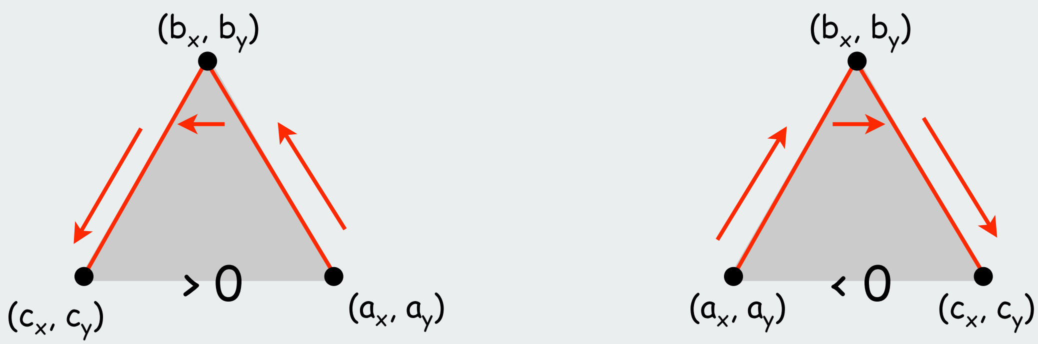
Implementing CCW

CCW. Given three point a, b, and c, is a-b-c a counterclockwise turn?

- Determinant gives twice area of triangle.

$$2 \times Area(a,b,c) = \begin{vmatrix} a_x & a_y & 1 \\ b_x & b_y & 1 \\ c_x & c_y & 1 \end{vmatrix} = (b_x - a_x)(c_y - a_y) - (b_y - a_y)(c_x - a_x)$$

- If area > 0 then a-b-c is counterclockwise.
- If area < 0, then a-b-c is clockwise.
- If area = 0, then a-b-c are collinear.



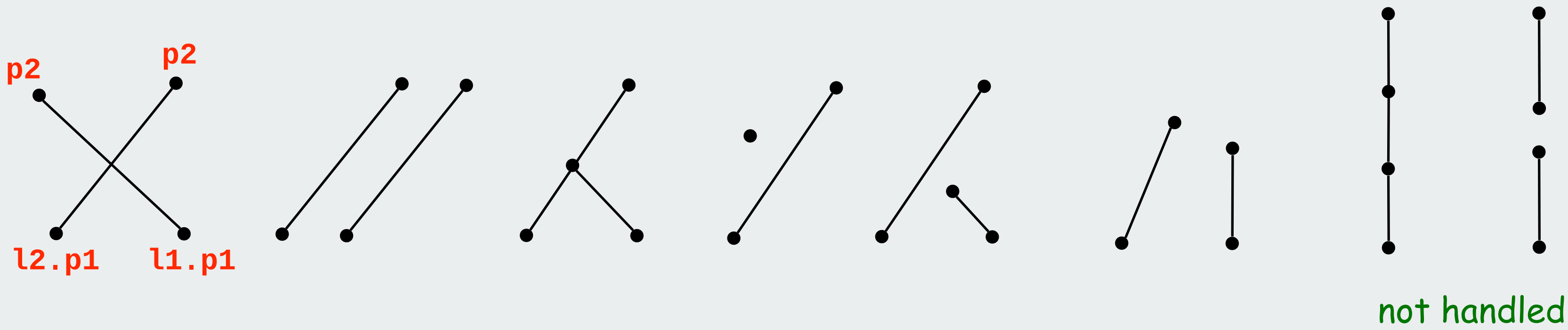
Immutable Point ADT

```
public final class Point
{
   public final int x;
   public final int y;

   public Point(int x, int y)
   {  this.x = x; this.y = y;  }


   public double distanceTo(Point q)
   {  return Math.hypot(this.x - q.x, this.y - q.y);  }

   public static int ccw(Point a, Point b, Point c)
   {
      double area2 = (b.x-a.x)*(c.y-a.y) - (b.y-a.y)*(c.x-a.x);
      if else (area2 < 0) return -1;
      else if (area2 > 0) return +1;
      else if (area2 > 0  return  0;
   }

   public static boolean collinear(Point a, Point b, Point c)
   {
      return ccw(a, b, c) == 0;
   }
}
```

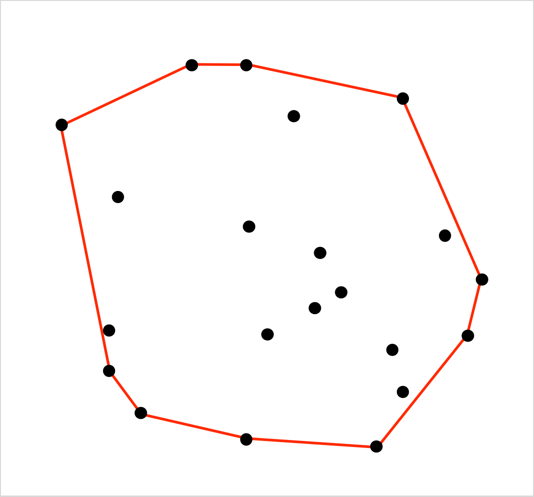
Sample ccw client: Line intersection

Intersect: Given two line segments, do they intersect?

- Idea 1: find intersection point using algebra and check.
- Idea 2: check if the endpoints of one line segment are on different "sides" of the other line segment.
- 4 ccw computations.




```
public static boolean intersect(Line l1, Line l2)
{
   int test1, test2;
   test1 = Point.ccw(l1.p1, l1.p2, l2.p1)
           * Point.ccw(l1.p1, l1.p2, l2.p2);
   test2 = Point.ccw(l2.p1, l2.p2, l1.p1)
           * Point.ccw(l2.p1, l2.p2, l1.p2);
   return (test1 <= 0) && (test2 <= 0);
}
```

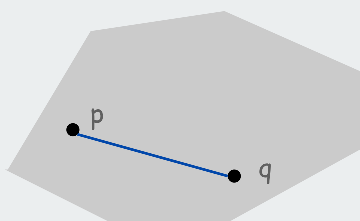


- primitive operations
- **convex hull**
- closest pair
- voronoi diagram

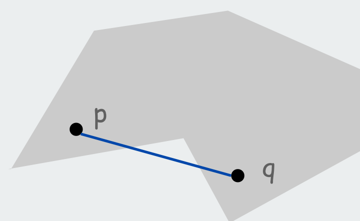
Convex Hull

A set of points is convex if for any two points p and q in the set, the line segment $\overline{pq}$ is completely in the set.

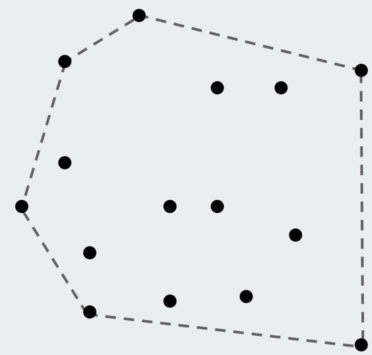
Convex hull. Smallest convex set containing all the points.



convex



not convex



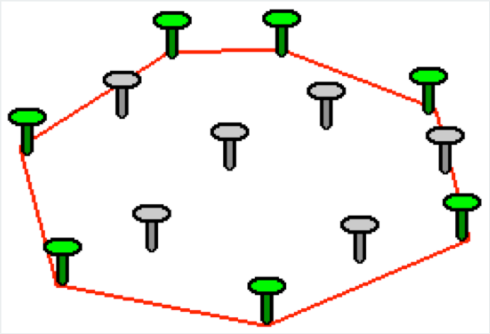
convex hull

Properties.

- "Simplest" shape that approximates set of points.
- Shortest (perimeter) fence surrounding the points.
- Smallest (area) convex polygon enclosing the points.

Mechanical Solution

Mechanical algorithm. Hammer nails perpendicular to plane; stretch elastic rubber band around points.



http://www.dfanning.com/math_tips/convexhull_1.gif

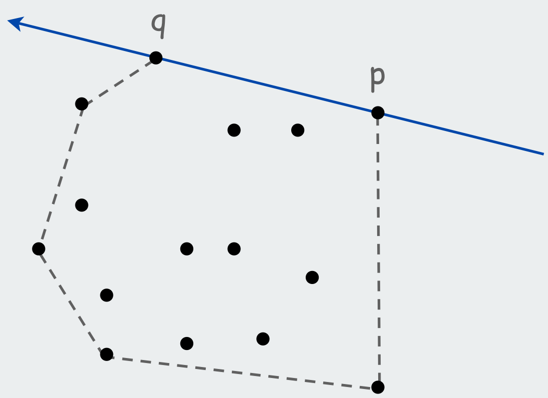
Brute-force algorithm

Observation 1.

Edges of convex hull of P connect pairs of points in P.

Observation 2.

p-q is on convex hull if all other points are counterclockwise of $\overrightarrow{pq}$.



$O(N^3)$ algorithm.

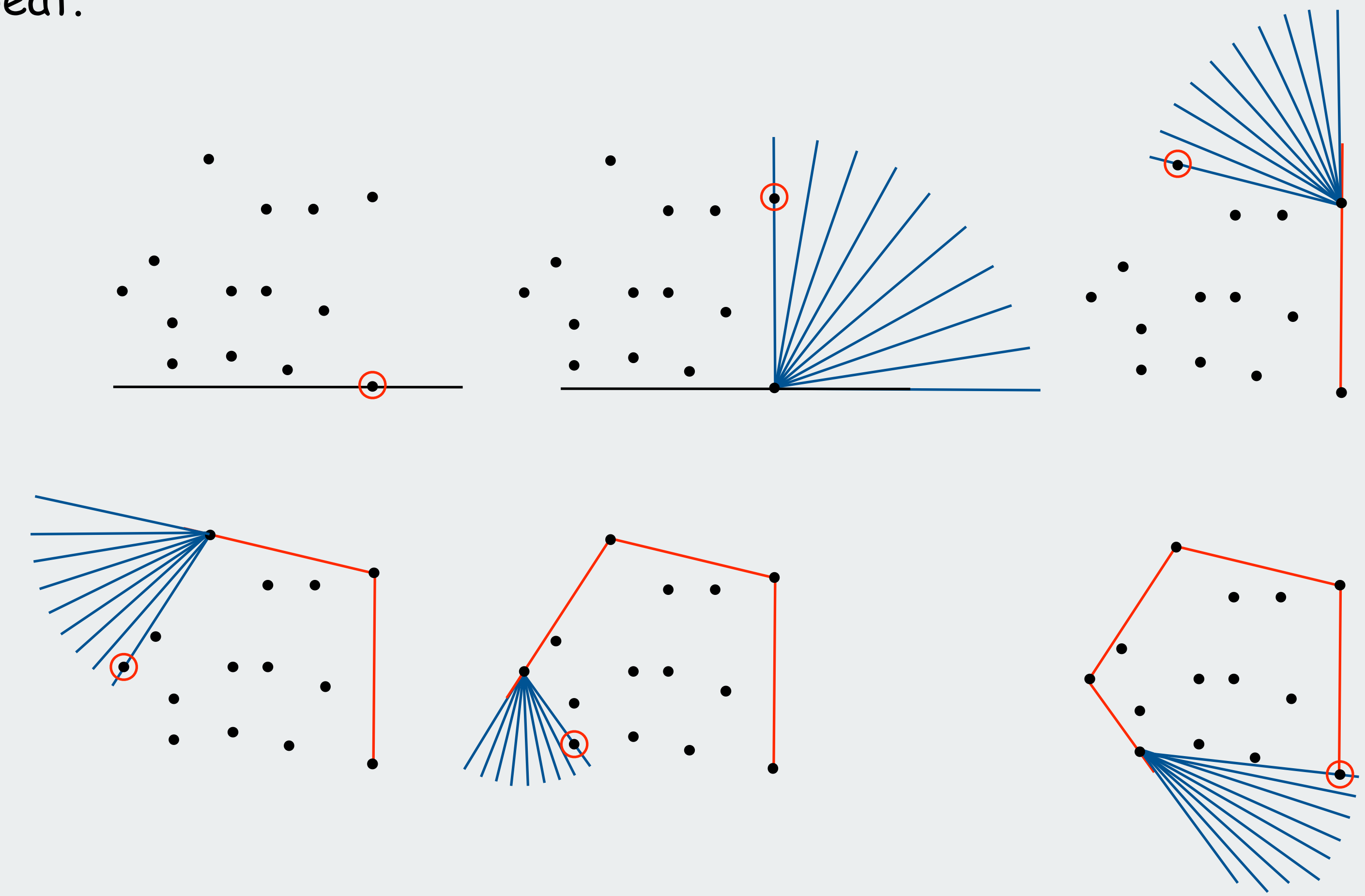
For all pairs of points p and q in P

- compute ccw(p, q, x) for all other x in P
- p-q is on hull if all values positive

Package Wrap (Jarvis March)

Package wrap.

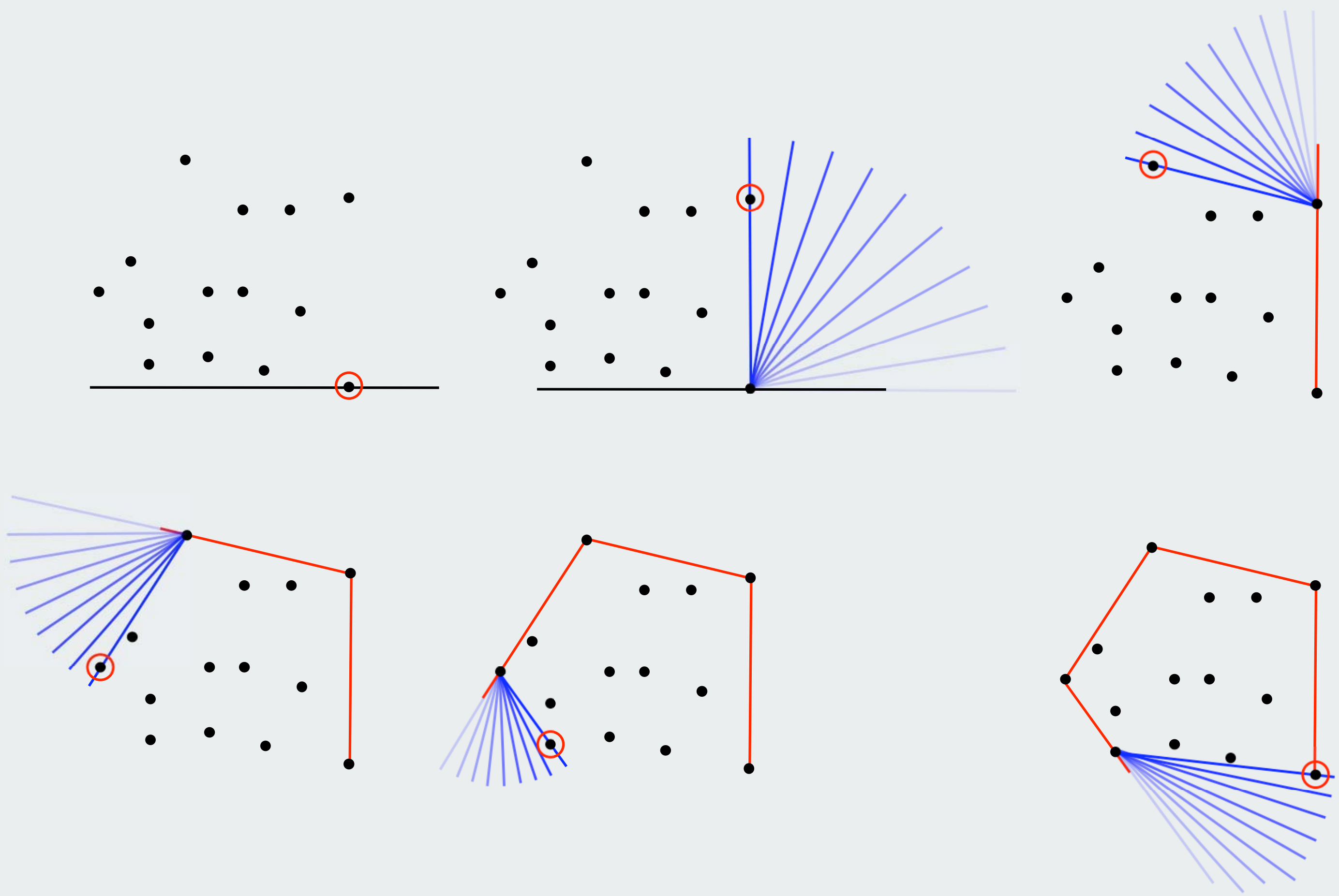
- Start with point with smallest y-coordinate.
- Rotate sweep line around current point in ccw direction.
- First point hit is on the hull.
- Repeat.



Package Wrap (Jarvis March)

Implementation.

- Compute angle between current point and all remaining points.
- Pick smallest angle larger than current angle.
- $\Theta(N)$ per iteration.



How Many Points on the Hull?

Parameters.

- N = number of points.
- h = number of points on the hull.

Package wrap running time. $\Theta(N\,h)$ per iteration.

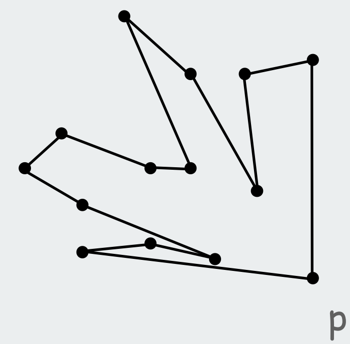
How many points on hull?

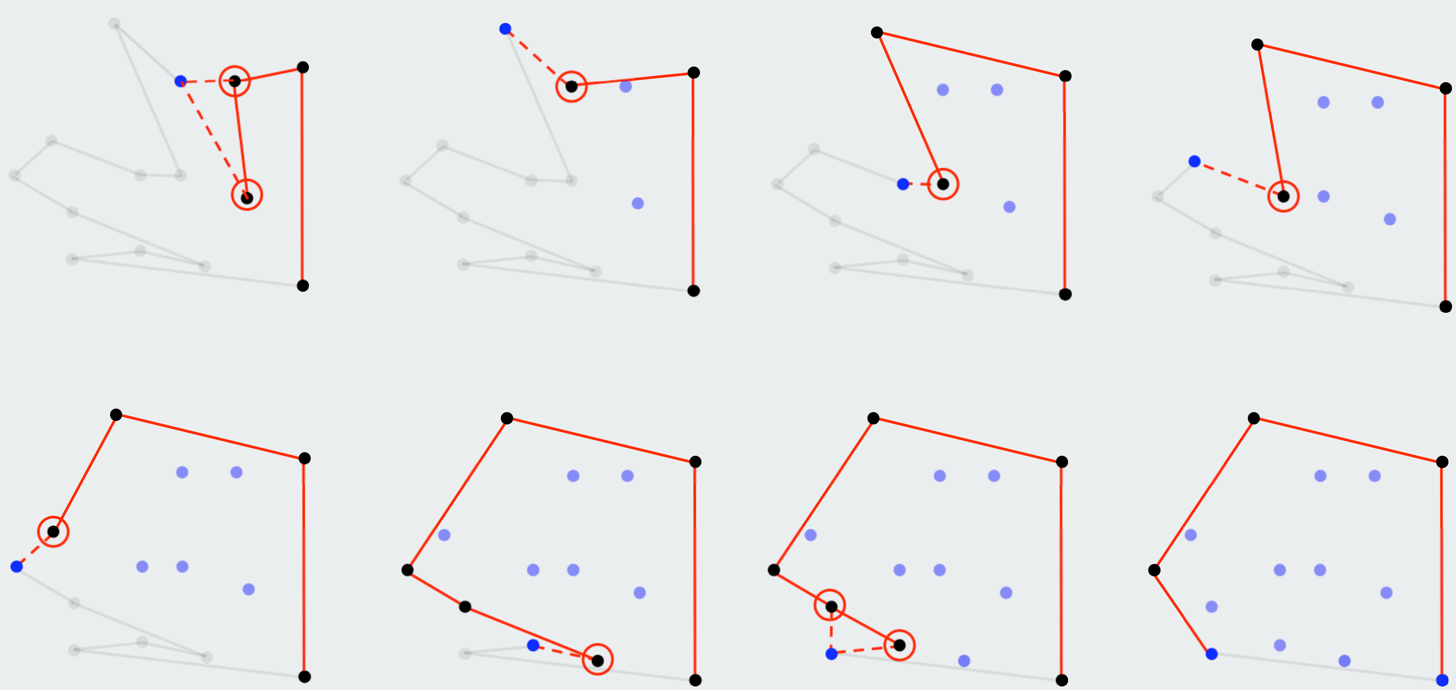
- Worst case: $h = N$.
- Average case: difficult problems in stochastic geometry.
  - in a disc: $h = N^{1/3}$.
  - in a convex polygon with $O(1)$ edges: $h = \log N$.

Graham Scan: Example

Graham scan.

- Choose point p with smallest y-coordinate.
- Sort points by polar angle with p to get simple polygon.
- Consider points in order, and discard those that would create a clockwise turn.





Graham Scan: Example

Implementation.

- Input: `p[1], p[2], …, p[N]` are points.
- Output: `M` and rearrangement so that `p[1],...,p[M]` is convex hull.

```
// preprocess so that p[1] has smallest y-coordinate
// sort by angle with p[1]

points[0] = points[N];  // sentinel
int M = 2;
for (int i = 3; i <= N; i++)
{
   while (Point.ccw(p[M-1], p[M], p[i]) <= 0) M--;
   M++;
   swap(points, M, i);
}
```

discard points that would create clockwise turn

add i to putative hull

Running time. O(N log N) for sort and O(N) for rest.

why?

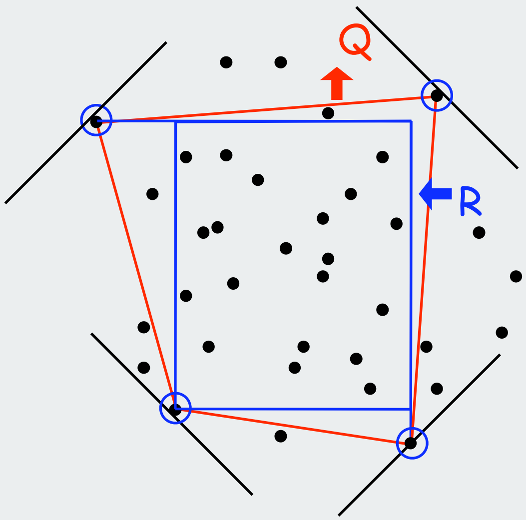
Quick Elimination

Quick elimination.

- Choose a quadrilateral Q or rectangle R with 4 points as corners.
- Any point inside cannot be on hull

  4 ccw tests for quadrilateral

  4 comparisons for rectangle

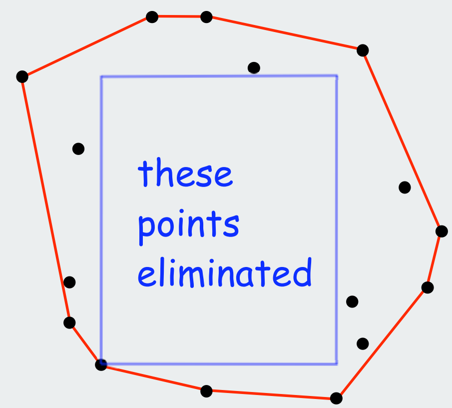


Three-phase algorithm

- Pass through all points to compute R.
- Eliminate points inside R.
- Find convex hull of remaining points.

In practice

can eliminate almost all points in linear time.



Convex Hull Algorithms Costs Summary

Asymptotic cost to find h-point hull in N-point set

algorithm	growth of running time
Package wrap	N h
Graham scan	N log N
Quickhull	N log N
Mergehull	N log N
Sweep line	N log N
Quick elimination	N †
Best in theory	N log h

output sensitive

† assumes "reasonable" point distribution

Convex Hull: Lower Bound

Models of computation.

- Comparison based: compare coordinates.
  (impossible to compute convex hull in this model of computation)

```
(a.x < b.x) || ((a.x == b.x) && (a.y < b.y)))
```

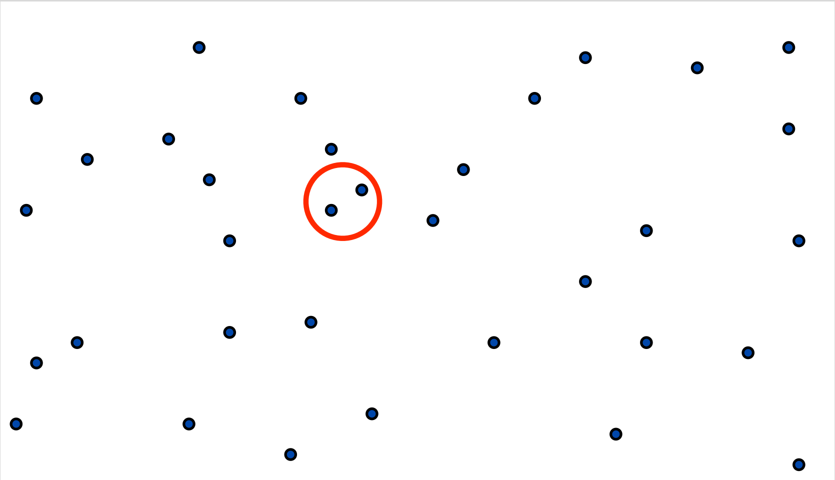
- Quadratic decision tree model: compute any quadratic function of the coordinates and compare against 0.

```
(a.x*b.y - a.y*b.x + a.y*c.x - a.x*c.y + b.x*c.y - c.x*b.y) < 0
```

Theorem. [Andy Yao, 1981] In quadratic decision tree model, any convex hull algorithm requires $\Omega(N \log N)$ ops.

even if hull points are not required to be output in counterclockwise order

higher degree polynomial tests don't help either [Ben-Or, 1983]



- primitive operations
- convex hull
- **closest pair**
- voronoi diagram

Closest pair problem

Given: N points in the plane

Goal: Find a pair with smallest Euclidean distance between them.

Fundamental geometric primitive.

- Graphics, computer vision, geographic information systems, molecular modeling, air traffic control.
- Special case of nearest neighbor, Euclidean MST, Voronoi.

fast closest pair inspired fast algorithms for these problems

Brute force.

Check all pairs of points p and q with $\Theta(N^2)$ distance calculations.

1-D version. O(N log N) easy if points are on a line.

as usual for geometric algs

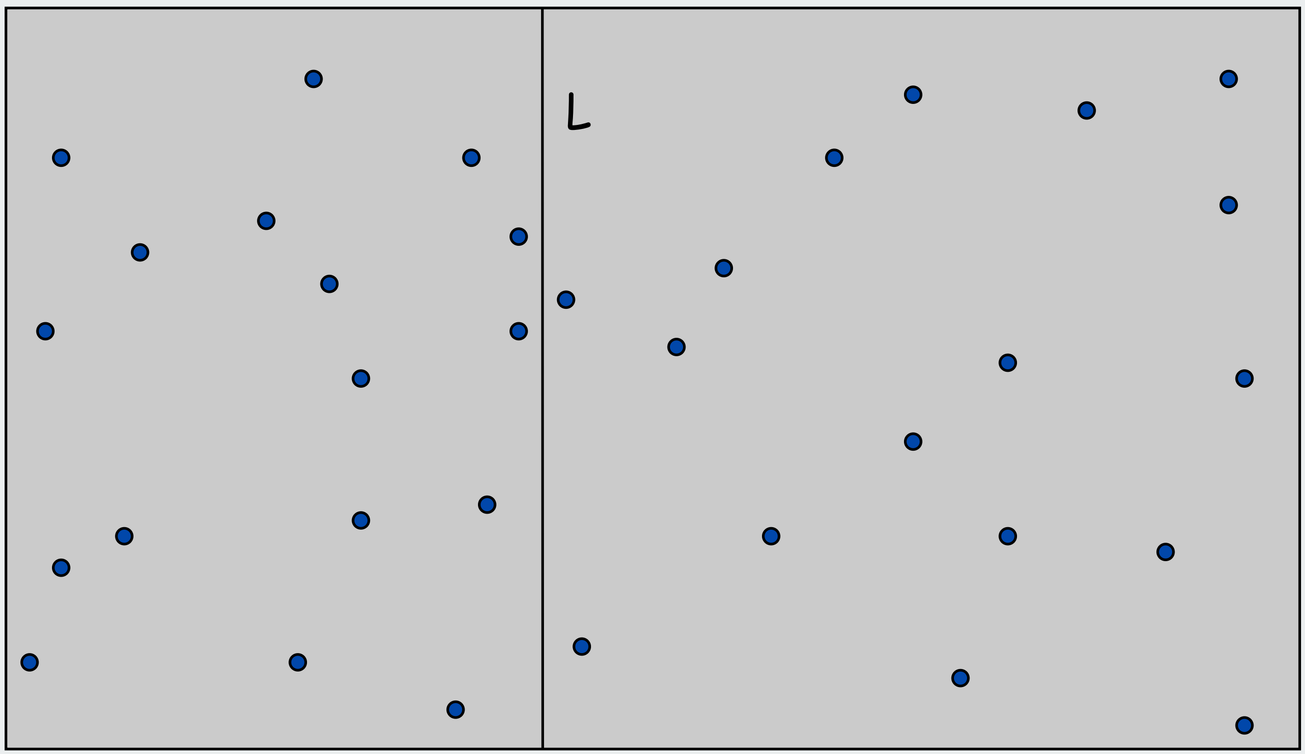
Degeneracies complicate solutions.

[assumption for lecture: no two points have same x coordinate]

Closest Pair of Points

Algorithm.

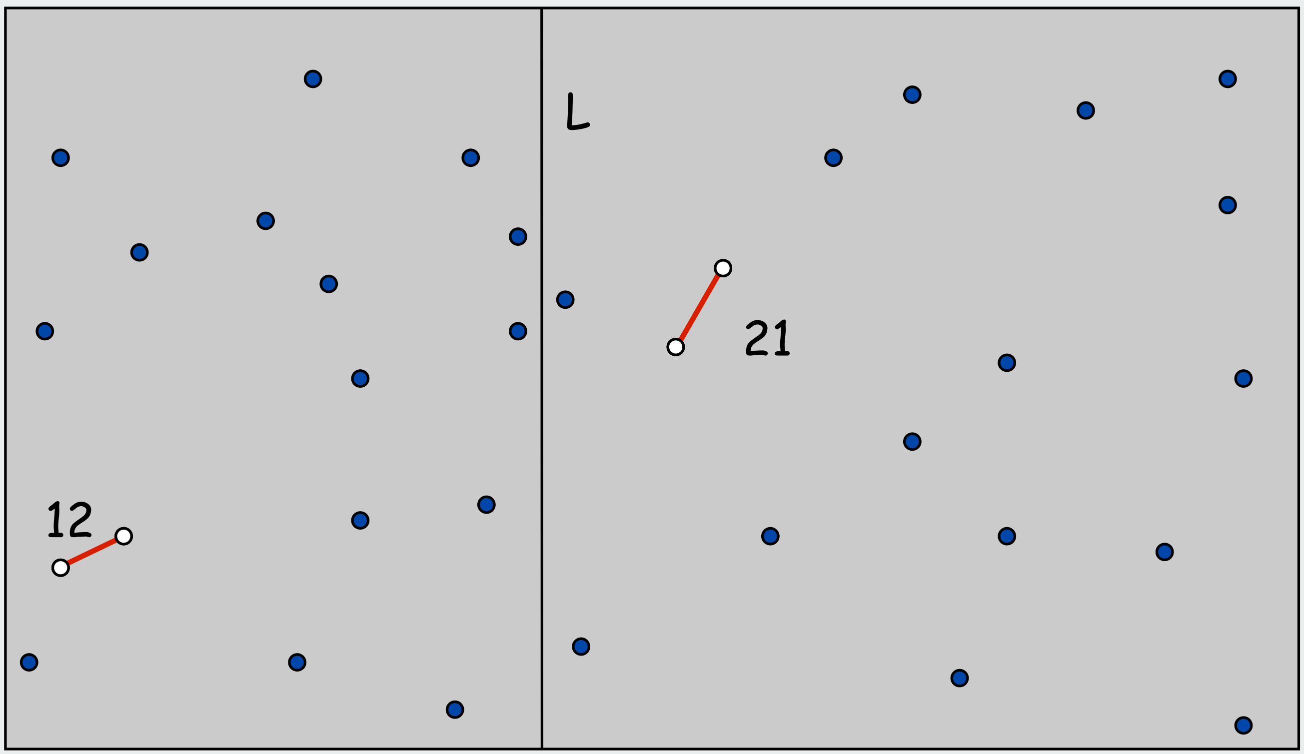
- Divide: draw vertical line L so that roughly $\frac{1}{2}N$ points on each side.



Closest Pair of Points

Algorithm.

- Divide: draw vertical line L so that roughly $\frac{1}{2}N$ points on each side.
- Conquer: find closest pair in each side recursively.

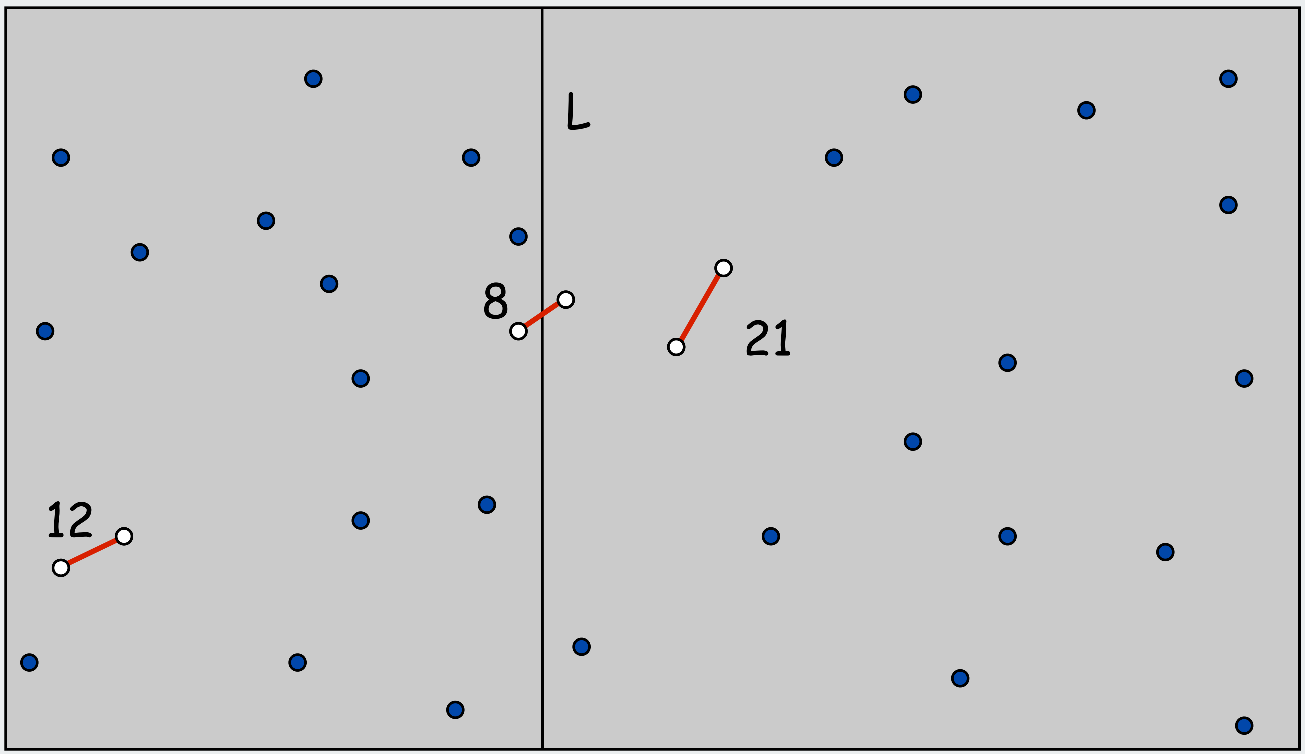


Closest Pair of Points

Algorithm.

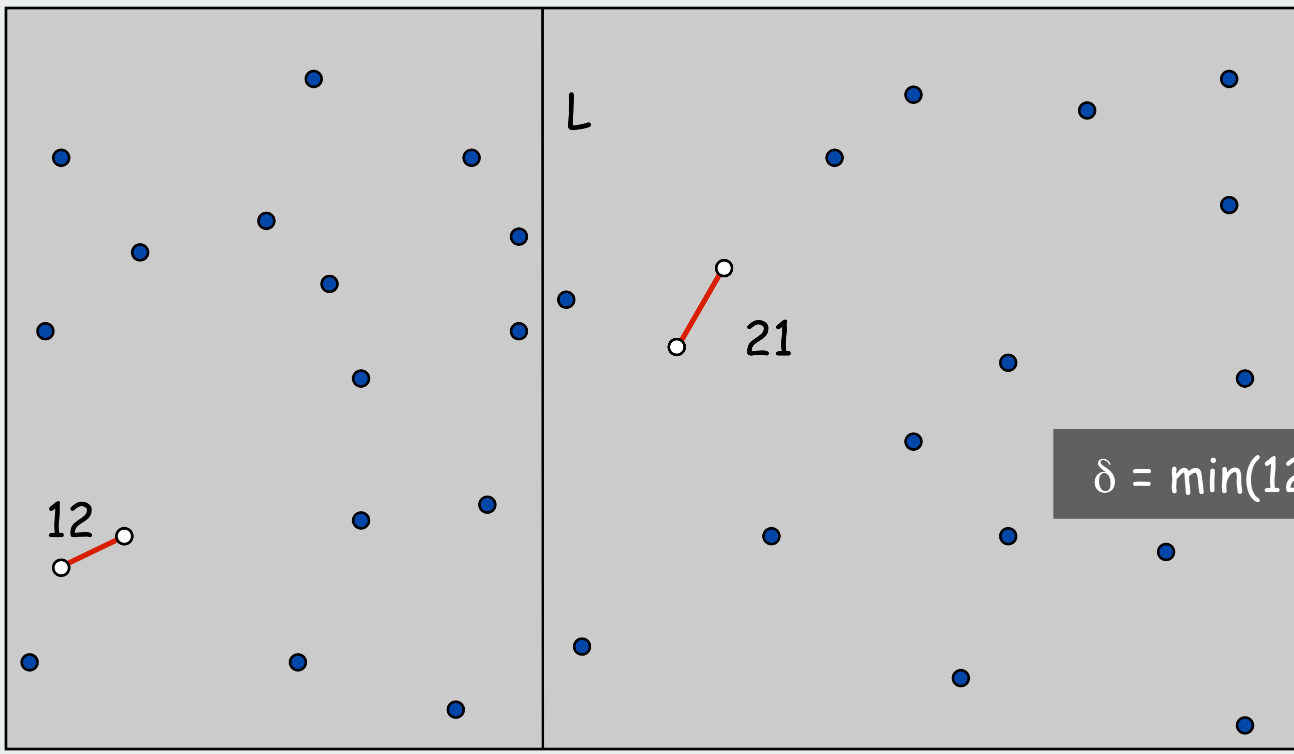
- Divide: draw vertical line L so that roughly $\frac{1}{2}N$ points on each side.
- Conquer: find closest pair in each side recursively.
- Combine: find closest pair with one point in each side.
- Return best of 3 solutions.

seems like $\Theta(N^2)$



Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < δ.

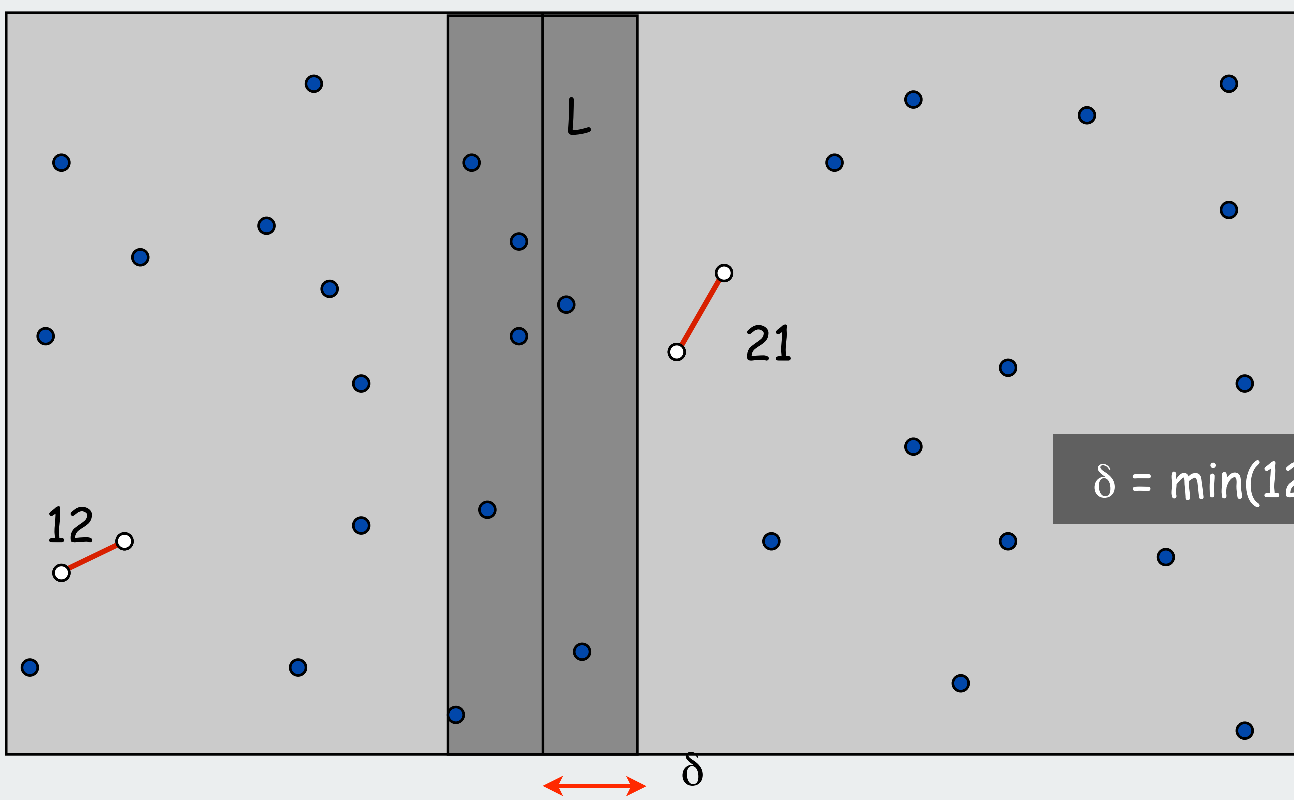


δ = min(12, 21)

Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < δ.

- Observation: only need to consider points within δ of line L.

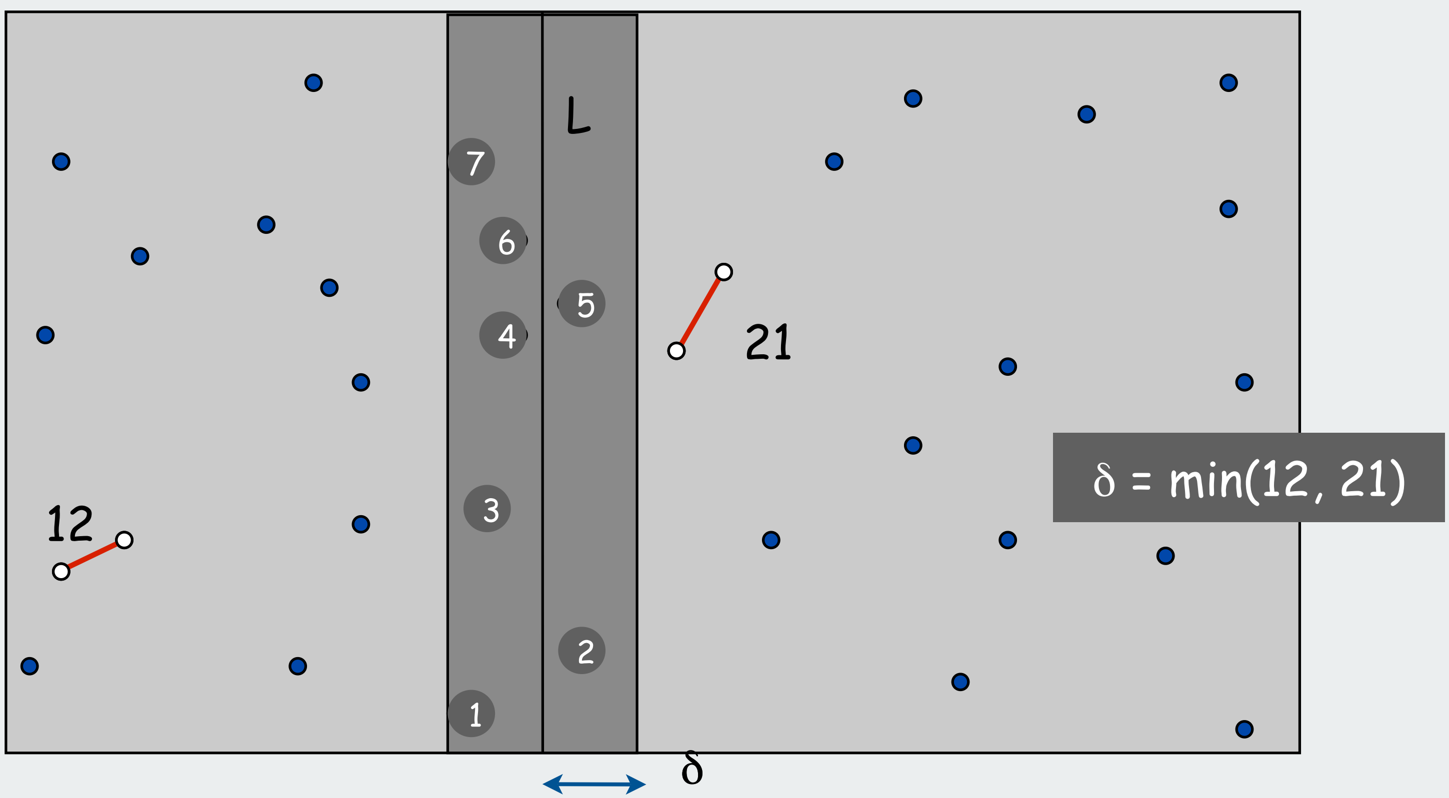


δ = min(12, 21)

Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < δ.

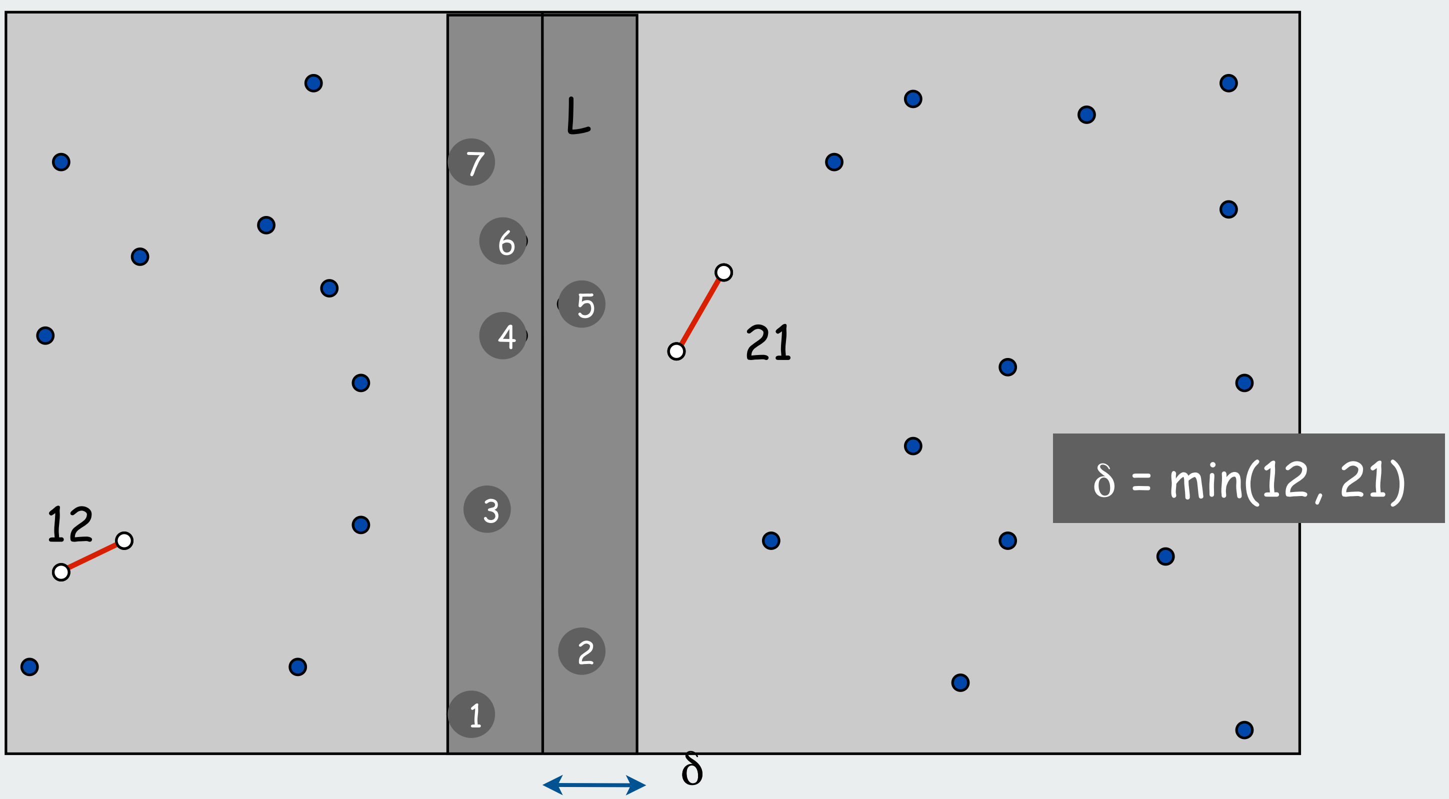
- Observation: only need to consider points within δ of line L.
- Sort points in 2δ-strip by their y coordinate.



Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < δ.

- Observation: only need to consider points within δ of line L.
- Sort points in 2δ-strip by their y coordinate.
- Only check distances of those within 11 positions in sorted list!



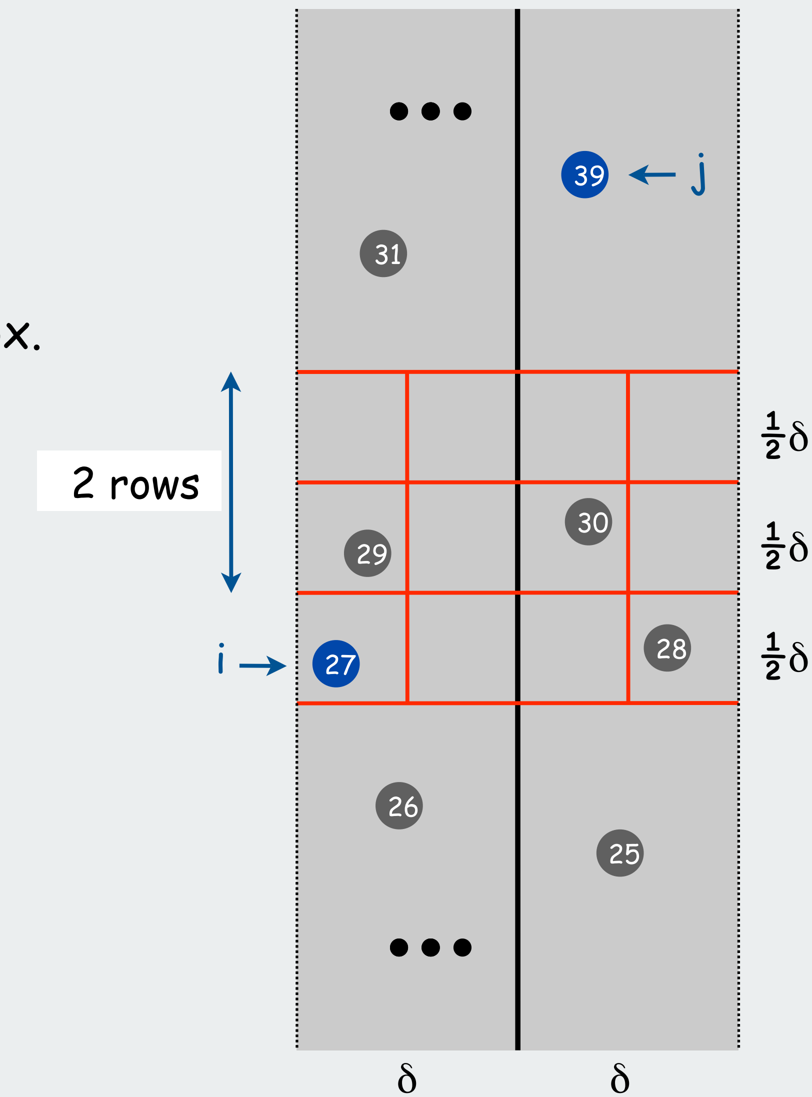
Def. Let s_i be the point in the 2δ-strip, with the i^{th} smallest y-coordinate.

Claim. If $|i - j| \geq 12$, then the distance between s_i and s_j is at least δ.

Pf.

- No two points lie in same $\frac{1}{2}\delta$-by-$\frac{1}{2}\delta$ box.
- Two points at least 2 rows apart have distance $\geq 2(\frac{1}{2}\delta)$. ▪

Fact. Still true if we replace 12 with 7.



Closest Pair Algorithm

```
Closest-Pair(p₁, …, pₙ)
{
   Compute separation line L such that half the points
   are on one side and half on the other side.                O(N log N)

   δ₁ = Closest-Pair(left half)
   δ₂ = Closest-Pair(right half)                               2T(N / 2)
   δ  = min(δ₁, δ₂)

   Delete all points further than δ from separation line L     O(N)

   Sort remaining points by y-coordinate.                      O(N log N)

   Scan points in y-order and compare distance between
   each point and next 11 neighbors. If any of these           O(N)
   distances is less than δ, update δ.

   return δ.
}
```

Closest Pair of Points: Analysis

Algorithm gives upper bound on running time

Recurrence

$$T(N) \le 2T(N/2) + O(N \log N)$$

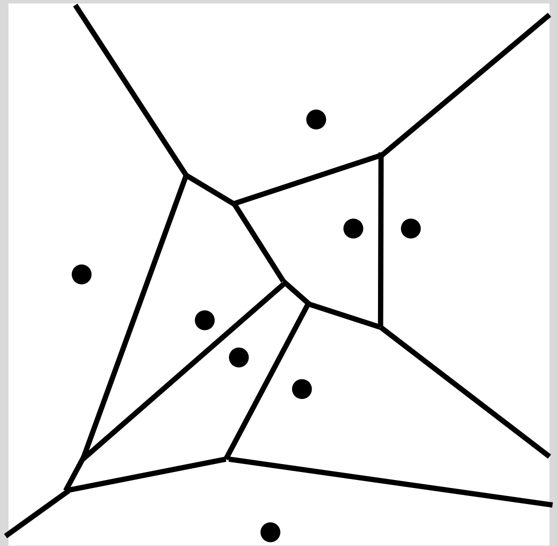
Solution

$$T(N) = O(N (\log N)^2)$$

avoid sorting by y-coordinate from scratch

Upper bound. Can be improved to $O(N \log N)$.

Lower bound. In quadratic decision tree model, any algorithm for closest pair requires $\Omega(N \log N)$ steps.



- primitive operations
- convex hull
- closest pair
- **voronoi diagrams**

1854 Cholera Outbreak, Golden Square, London

Life-or-death question:

Given a new cholera patient p, which water pump is closest to p's home?

http://content.answers.com/main/content/wp/en/c/c7/Snow-cholera-map.jpg

Nearest-neighbor problem

Input.
N Euclidean points.

Nearest neighbor problem.
Given a query point p, which one of original N points is closest to p?

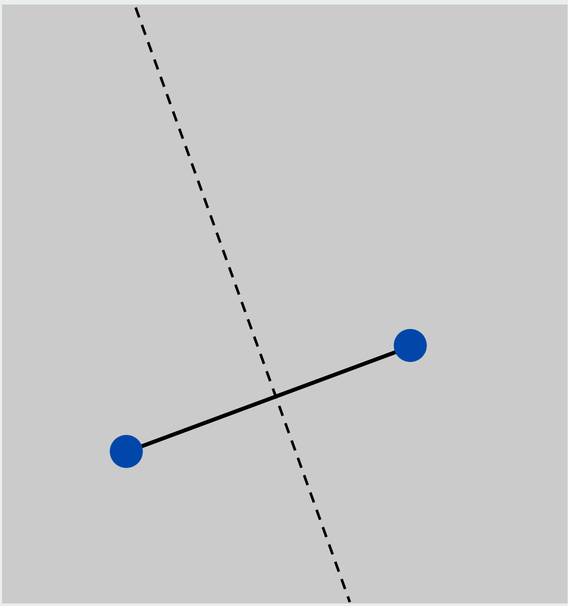
Algorithm	Preprocess	Query
Brute	1	N
Goal	N log N	log N

Voronoi Diagram

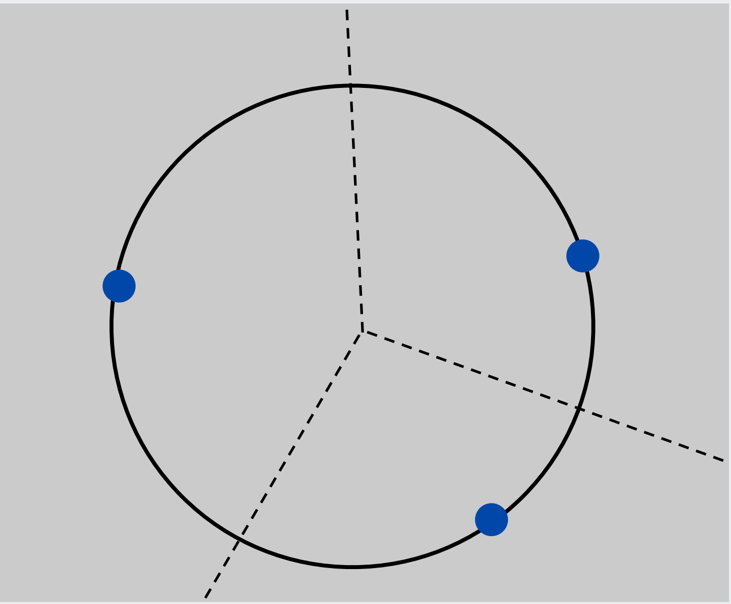
Voronoi region. Set of all points closest to a given point.

Voronoi diagram. Planar subdivision delineating Voronoi regions.

Fact. Voronoi edges are perpendicular bisector segments.



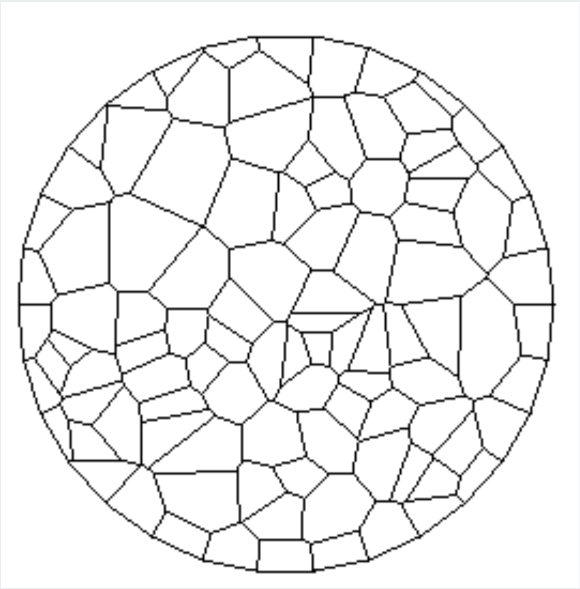
Voronoi of 2 points
(perpendicular bisector)

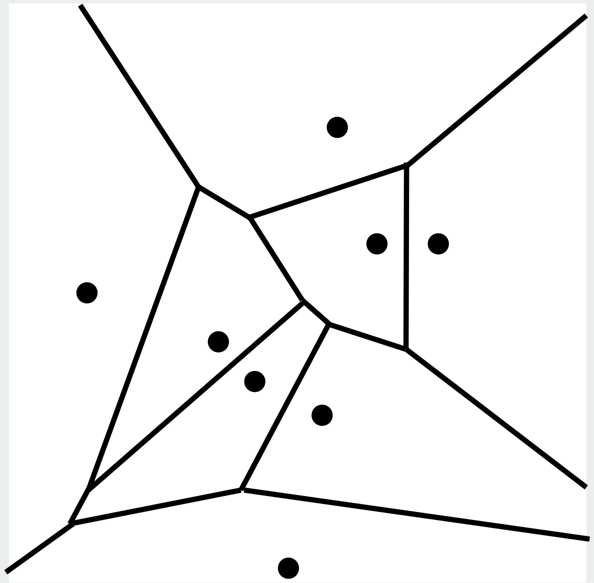


Voronoi of 3 points
(passes through circumcenter)

Voronoi Diagram

Voronoi region. Set of all points closest to a given point.
Voronoi diagram. Planar subdivision delineating Voronoi regions.
Fact. Voronoi edges are perpendicular bisector segments.





Quintessential nearest neighbor data structure.

Voronoi Diagram: Applications

Toxic waste dump problem. N homes in a region. Where to locate nuclear power plant so that it is far away from any home as possible?

looking for largest empty circle
(center must lie on Voronoi diagram)

Path planning. Circular robot must navigate through environment with N obstacle points. How to minimize risk of bumping into a obstacle?

robot should stay on Voronoi diagram of obstacles

Reference: J. O'Rourke. Computational Geometry.

Voronoi Diagram: More Applications

Anthropology. Identify influence of clans and chiefdoms on geographic regions.

Astronomy. Identify clusters of stars and clusters of galaxies.

Biology, Ecology, Forestry. Model and analyze plant competition.

Cartography. Piece together satellite photographs into large "mosaic" maps.

Crystallography. Study Wigner-Setiz regions of metallic sodium.

Data visualization. Nearest neighbor interpolation of 2D data.

Finite elements. Generating finite element meshes which avoid small angles.

Fluid dynamics. Vortex methods for inviscid incompressible 2D fluid flow.

Geology. Estimation of ore reserves in a deposit using info from bore holes.

Geo-scientific modeling. Reconstruct 3D geometric figures from points.

Marketing. Model market of US metro area at individual retail store level.

Metallurgy. Modeling "grain growth" in metal films.

Physiology. Analysis of capillary distribution in cross-sections of muscle tissue.

Robotics. Path planning for robot to minimize risk of collision.

Typography. Character recognition, beveled and carved lettering.

Zoology. Model and analyze the territories of animals.

References: http://voronoi.com, http://www.ics.uci.edu/~eppstein/geom.html

Scientific Rediscoveries

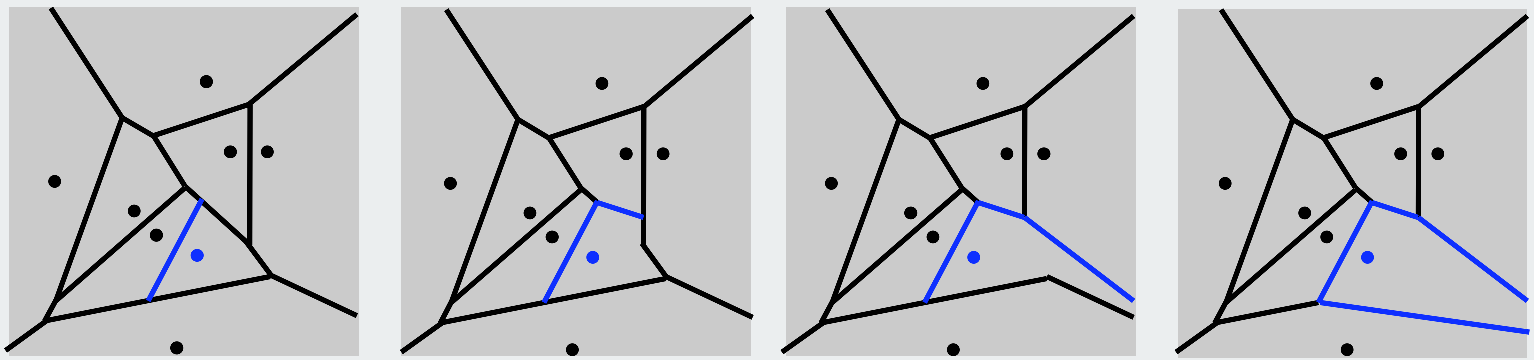
Year	Discoverer	Discipline	Name
1644	Descartes	Astronomy	"Heavens"
1850	Dirichlet	Math	Dirichlet tesselation
1908	Voronoi	Math	Voronoi diagram
1909	Boldyrev	Geology	area of influence polygons
1911	Thiessen	Meteorology	Thiessen polygons
1927	Niggli	Crystallography	domains of action
1933	Wigner-Seitz	Physics	Wigner-Seitz regions
1958	Frank-Casper	Physics	atom domains
1965	Brown	Ecology	area of potentially available
1966	Mead	Ecology	plant polygons
1985	Hoofd et al.	Anatomy	capillary domains

Reference: Kenneth E. Hoff III

Adding a Point to Voronoi Diagram

Challenge. Compute Voronoi.

Basis for incremental algorithms: region containing point gives points to check to compute new Voronoi region boundaries.



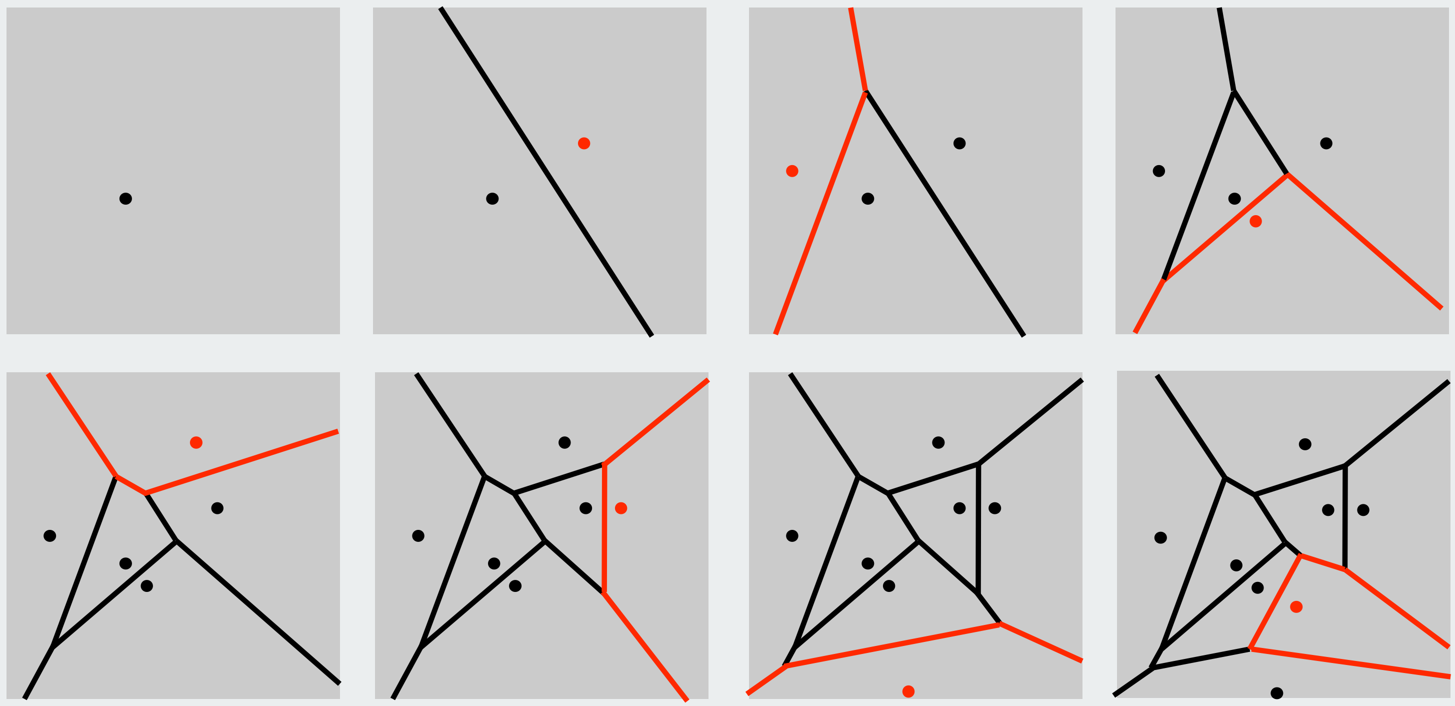
How to represent the Voronoi diagram?
Use multilist associating each point with its Voronoi neighbors

How to find region containing point?
Use Voronoi itself (possible, but not easy!)

Randomized Incremental Voronoi Algorithm

Add points (in random order).

- Find region containing point. ← using Voronoi itself
- Update neighbor regions, create region for new point.

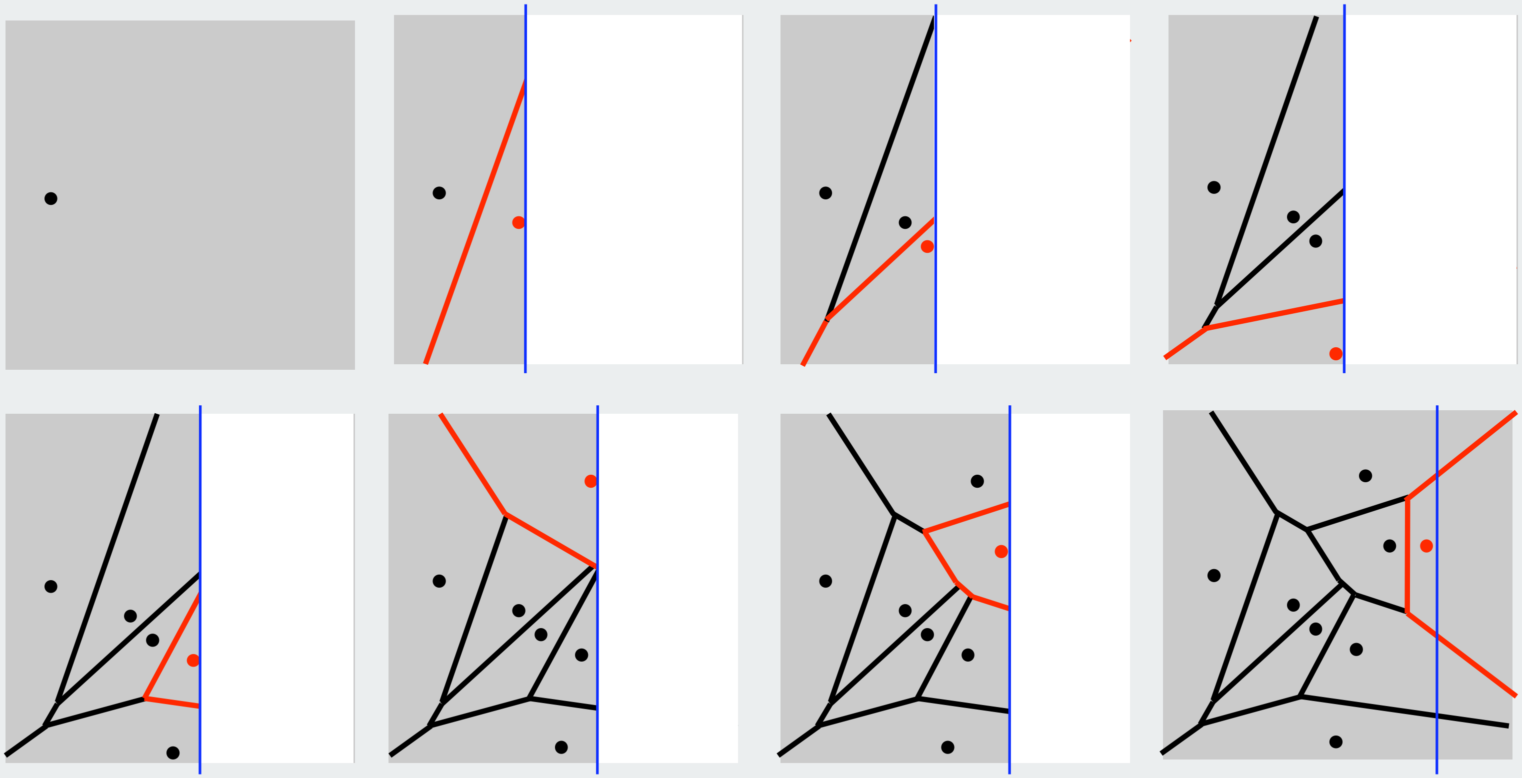


- Running time: O(N log N) on average.

Not an elementary algortihm

Sweep-line Voronoi algorithm

Presort points on x-coordinate
Eliminates point location problem



Fortune's Algorithm

Industrial-strength Voronoi implementation.

- Sweep-line algorithm
- O(N log N) time
- properly handles degeneracies
- properly handles floating-point computations

Algorithm	Preprocess	Query
Brute	1	N
Goal	N log N	log N

Try it yourself!

http://www.diku.dk/hjemmesider/studerende/duff/Fortune/

best animation on the web
student Java project
"lost" the source
decompiled source available

Interface between numeric and combinatorial computing

- exact calculations impossible (using floating point)
- exact calculations required!
- one solution: randomly jiggle the points

Fortune's algorithm in action

http://www.diku.dk/hjemmesider/studerende/duff/Fortune/

Fortune's algorithm in action

Fortune's algorithm in action

Fortune's algorithm in action

Fortune's algorithm in action

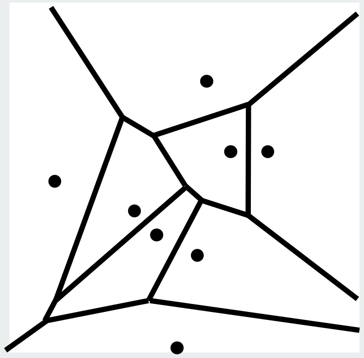
Geometric-algorithm challenge

Problem: Draw a Voronoi diagram

Goals: lecture slide, book diagram

How difficult?

1) any COS126 student could do it
2) need to be a typical diligent COS226 student
3) hire an expert
4) intractable
5) no one knows
6) impossible



Geometric-algorithm challenge

Problem: Draw a Voronoi diagram

Goals: lecture slide, book diagram

surprise!

How difficult?

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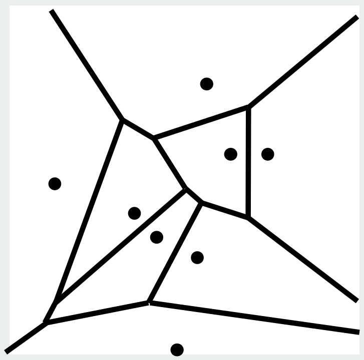
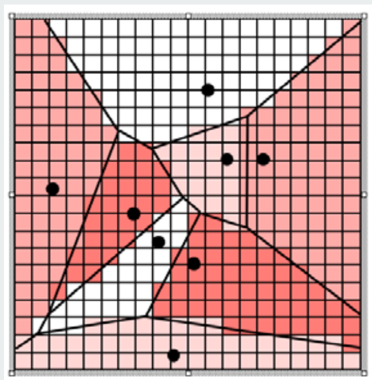


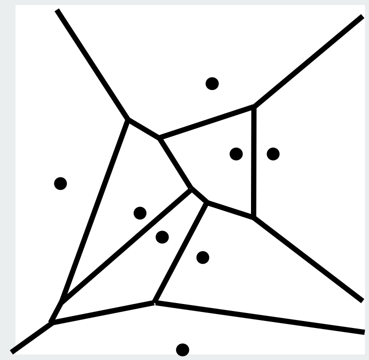
Discretized Voronoi diagram

Observation: to draw a Voronoi diagram, only need an approximation

Ex: Assign a color to each pixel corresponding to its nearest neighbor



P pixels



An effective approximate solution to the nearest neighbor problem

Algorithm	Preprocess	Query
Brute	1	N
Fortune	N log N	log N
Discretized	N P	1

complicated alg (stay tuned)

Discretized Voronoi: Java Implementation

InteractiveDraw. Version of **StdDraw** that supports user interaction.

DrawListener. Interface to support **InteractiveDraw** callbacks.

```
public class Voronoi implements DrawListener
{
   private int SIZE = 512;
   private Point[][] nearest = new Point[SIZE][SIZE];
   private InteractiveDraw draw;
   public Voronoi()
   {
      draw = new InteractiveDraw(SIZE, SIZE);
      draw.setScale(0, 0, SIZE, SIZE);
      draw.addListener(this);
      draw.show();
   }

   public void keyTyped(char c) { }
   public void mouseDragged (double x, double y) { }
   public void mouseReleased(double x, double y) { }
   public void mousePressed
   {  /* See next slide  */  }

}
```

send callbacks to **Voronoi**

http://www.cs.princeton.edu/introcs/35inheritance/Voronoi.java

Discretized Voronoi: Java Implementation

```
public void mousePressed(double x, double y)
{
   Point p = new Point(x, y);
   draw.setColorRandom();
   for (int i = 0; i < SIZE; i++)
      for (int j = 0; j < SIZE; j++)
      {
         Point q = new Point(i, j);
         if ((nearest[i][j] == null) ||
             (q.distanceTo(p) < q.distanceTo(nearest[i][j])))
         {
            nearest[i][j] = p;
            draw.moveTo(i, j);
            draw.spot();
         }
      }
   draw.setColor(StdDraw.BLACK);
   draw.moveTo(x, y);
   draw.spot(4);
   draw.show();
}
```

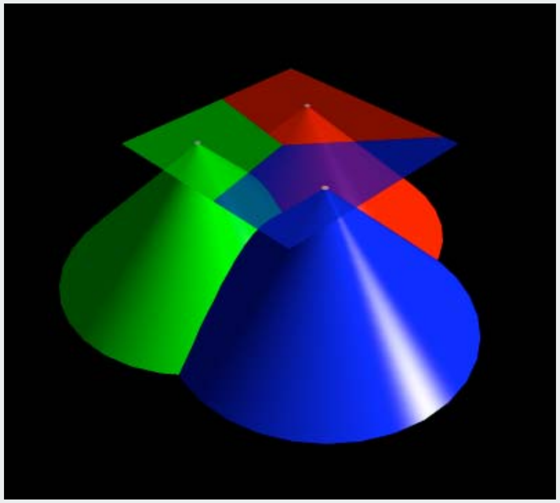
user clicks (x, y)

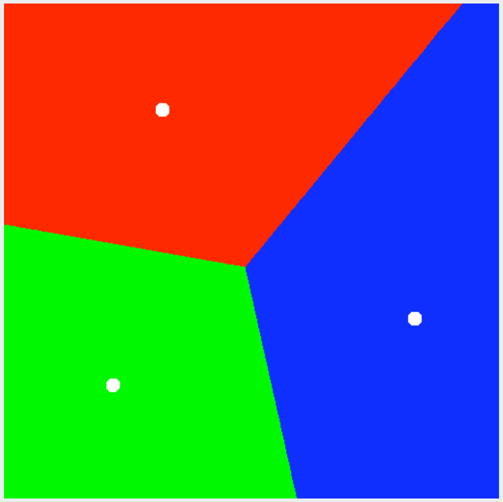
check every other point q to see if p became its nearest neighbor

Voronoi alternative 2: Hoff's algorithm

Hoff's algorithm. Align apex of a right circular cone with sites.

- Minimum envelope of cone intersections projected onto plane is the Voronoi diagram.
- View cones in different colors $\Rightarrow$ render Voronoi.

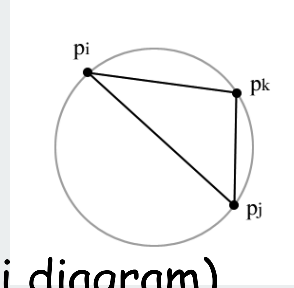




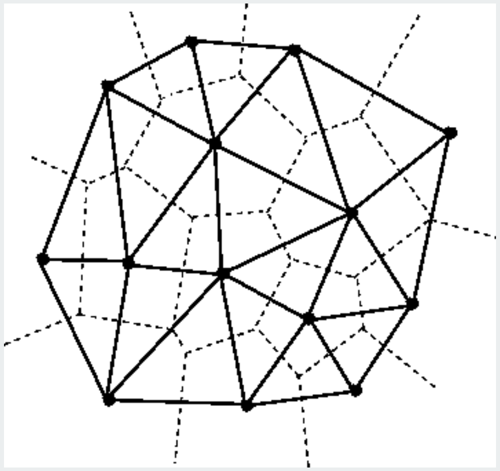
Implementation. Draw cones using standard graphics hardware!

http://www.cs.unc.edu/~geom/voronoi/siggraph_paper/voronoi.pdf

Delaunay triangulation. Triangulation of N points such that no point is inside circumcircle of any other triangle.



Fact 0. It exists and is unique (assuming no degeneracy).
Fact 1. Dual of Voronoi (connect adjacent points in Voronoi diagram).
Fact 2. No edges cross $\Rightarrow$ O(N) edges.
Fact 3. Maximizes the minimum angle for all triangular elements.
Fact 4. Boundary of Delaunay triangulation is convex hull.
Fact 5. Shortest Delaunay edge connects closest pair of points.

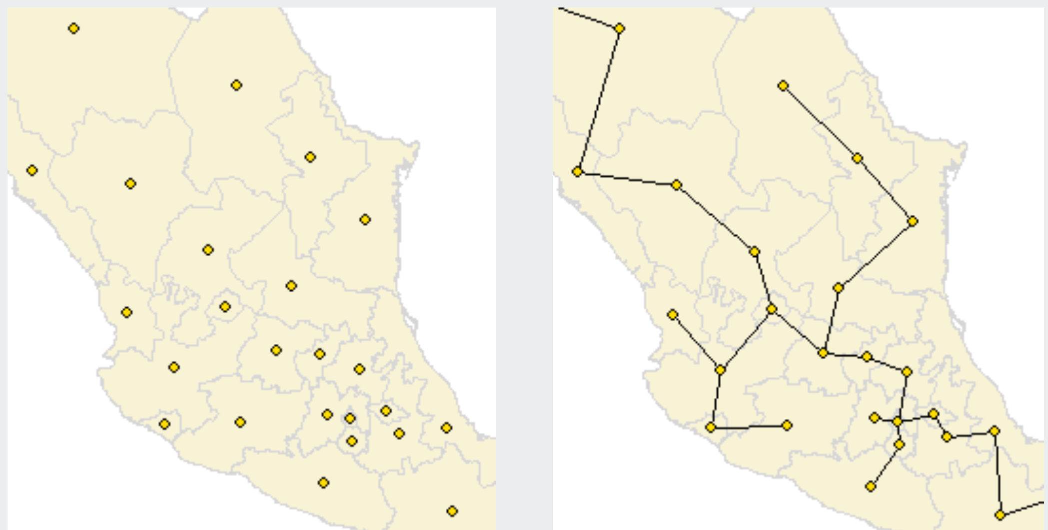


Delaunay
Voronoi

Euclidean MST

Euclidean MST. Given N points in the plane, find MST connecting them.

- Distances between point pairs are Euclidean distances.



Brute force. Compute $N^2 / 2$ distances and run Prim's algorithm.

Ingenuity.

- MST is subgraph of Delauney triagulation
- Delauney has O(N) edges
- Compute Delauney, then use Prim or Kruskal to get MST in O(N log N) !

Summary

Ingenuity in algorithm design can enable solution
of large instances for numerous fundamental geometric problems.

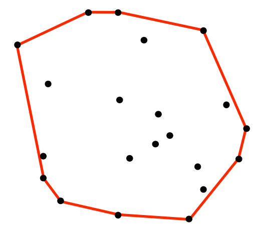
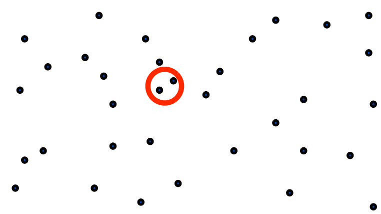
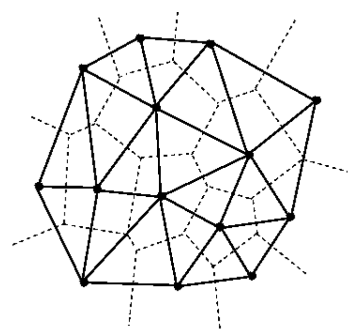
Problem	Brute	Cleverness
convex hull	N^2	N log N
closest pair	N^2	N log N
Voronoi	?	N log N
Delaunay triangulation	N^4	N log N
Euclidean MST	N^2	N log N

asymptotic time to solve a 2D problem with N points

Note: 3D and higher dimensions test limits of our ingenuity

Geometric algorithms summary: Algorithms of the day

asymptotic time to solve a 2D problem with N points

		brute	ingenuity
convex hull		N^2	N log N
closest pair		N^2	N log N
Voronoi/Delauney		N^4	N log N
Euclidean MST		N^2	N log N