Algorithm Design and Analysis

LECTURE 6

Greedy Algorithms

- Problems
  - Interval Scheduling
  - Interval Partitioning
- Analysis
  - Greedy Stays Ahead
  - Structural Argument

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Review Question from Lecture 4

- Give an example of functions f and g such that $f(n)=O(g(n))$ and $f(n)>g(n)$ for all $n\geq 1$.

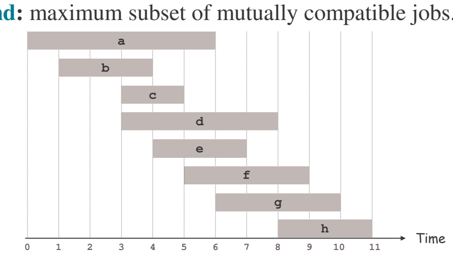
(Answer: $f(n)=2n$; $g(n)=n$)

Greedy Algorithms

- Build up a solution to an optimization problem at each step shortsightedly choosing the option that currently seems the best.
  - Sometimes good
  - Often does not work

Interval Scheduling Problem

- Job j starts at s_j and finishes at f_j.
- Two jobs are **compatible** if they do not overlap.
- **Find:** maximum subset of mutually compatible jobs.



Possible Greedy Strategies

Consider jobs in some natural order. Take next job if it is compatible with the ones already taken.

- **Earliest start time:** ascending order of s_j.
- **Earliest finish time:** ascending order of f_j.
- **Shortest interval:** ascending order of $(f_j - s_j)$.
- **Fewest conflicts:** For each job j, count the number of conflicting jobs c_j. Schedule in ascending order of c_j.

Greedy: Counterexamples

for earliest start time

for shortest interval


for fewest conflicts

Greedy Algorithm

- **Earliest finish time:** ascending order of f_j.

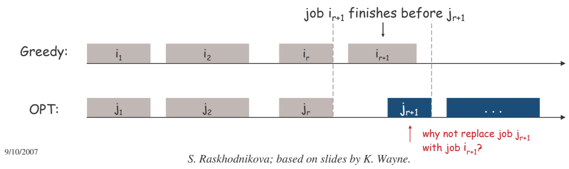
```
Sort jobs by finish times so that f1 ≤ f2 ≤ ... ≤ fn.

A ← φ     Δ Set of selected jobs
for j = 1 to n {
   if (job j compatible with A)
      A ← A ∪ {j}
}
return A
```

- Implementation. **O(n log n) time; O(n) space.**
  - Remember job j* that was added last to A.
  - Job j is compatible with A if $s_j \geq f_{j^*}$.

Analysis: Greedy Stays Ahead

- Theorem. Greedy algorithm is optimal.
- Pf (by contradiction): Suppose greedy is not optimal.
  - Let $i_1, i_2, \ldots i_k$ denote set of jobs selected by greedy.
  - Let $j_1, j_2, \ldots j_m$ be set of jobs in the optimal solution with $i_1 = j_1, i_2 = j_2, \ldots, i_r = j_r$ for the largest possible value of r.



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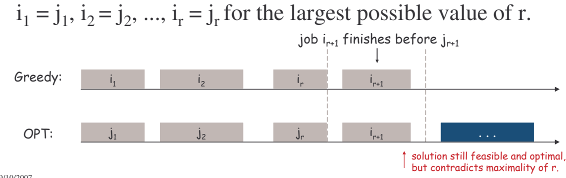
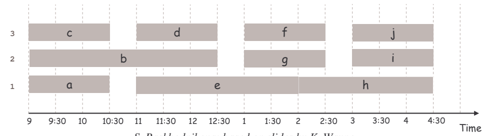


Interval Partitioning Problem

- Lecture j starts at s_j and finishes at f_j.
- **Find**: minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.
- **E.g.**: 10 lectures are scheduled in **4** classrooms.

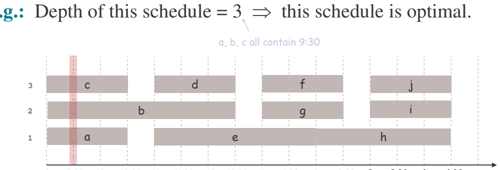
Interval Partitioning

- Lecture j starts at s_j and finishes at f_j.
- **Find**: minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.
- **E.g.**: Same lectures are scheduled in **3** classrooms.



Lower Bound

- **Definition**. The **depth** of a set of open intervals is the maximum number that contain any given time.
- **Key observation**. Number of classrooms needed $\geq$ depth.
- **E.g.:** Depth of this schedule = 3 $\Rightarrow$ this schedule is optimal.



- **Q:** Is it always sufficient to have number of classrooms = depth?

Greedy Algorithm

- Consider lectures in increasing order of start time: assign lecture to any compatible classroom.

```
Sort intervals by starting time so that s1 ≤ s2 ≤ ... ≤ sn.
d ← 0   Δ Number of allocated classrooms
for j = 1 to n {
   if (lecture j is compatible with some classroom k)
      schedule lecture j in classroom k
   else
      allocate a new classroom d + 1
      schedule lecture j in classroom d + 1
      d ← d + 1
}
```

- Implementation. O(n log n) time; O(n) space.
  - For each classroom, maintain the finish time of the last job added.
  - Keep the classrooms in a priority queue.

Analysis: Structural Argument

- **Observation**. Greedy algorithm never schedules two incompatible lectures in the same classroom.
- **Theorem**. Greedy algorithm is optimal.
- **Proof:** Let d = number of classrooms allocated by greedy.
  - Classroom d is opened because we needed to schedule a lecture, say j, that is incompatible with all d-1 last lectures in other classrooms.
  - These d lectures each end after s_j.
  - Since we sorted by start time, they start no later than s_j.
  - Thus, we have d lectures overlapping at time $s_j + \varepsilon$.
  - Key observation $\Rightarrow$ all schedules use $\geq$ d classrooms. ▪